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LITERATURE REVIEW

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Behavioral fluency and mathematics intervention research: A review of the last 20 years

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Behavioral fluency refers to the relationship between the achievement of performance standards, or frequency ranges of behavior, and critical learning outcomes. Over the past 20 years, Precision Teaching and related research have contributed a number of studies examining behavioral fluency. The subsequent review investigates the empirical evidence from mathematics intervention research. Several studies suggest numerical markers that best support behavioral fluency. Results indicate that fluency interventions set to performance standards increased behavioral fluency and associated critical learning outcomes; however, more research is warranted to operationalize and standardize each outcome to the principles of behavior and numerical markers that constitute behavioral fluency.

KEYWORDS

behavioral fluency, mathematics fluency, mathematics interventions, mathematics standards, precision teaching

1 | INTRODUCTION

Mathematics permeates many aspects of successful 21st century living. The link between mathematics and science fuels progress; citizens who can reason and communicate mathematically and scientifically have greater opportunities to understand and contribute to the social, political, and economic structures of modern society (Geary, 2013; Gross, Hudson, & Price, 2009). Conventional skills such as counting money, telling time, estimation, and basic problem solving remain necessary for successful daily living. Managing health care and finances in the new economy as well as acclimating to new technologies and related applications will require advanced mathematical skill sets (Lockard & Wolf, 2012; Price & Ansari, 2013). Moreover, to meet the demands of a burgeoning scientific and technological workforce, current students must acquire key competencies in mathematics. Nearly two thirds of entry-level jobs in the coming decade will require at least some advanced knowledge in algebra, statistics, data interpretation, and geometry to support gainful employment (Friedman, 2007; Lockard & Wolf, 2012).

National initiatives consider computational and procedural fluency as key components to mathematical competency (National Council of Teachers of Mathematics, NCTM, 2000, 2006, 2014; National Mathematics Advisory Panel, NMAP, 2008). Both NCTM (2000) and NMAP (2008) emphasize the importance of accuracy, automatic execution, and flexible use of whole number operations and standard algorithms, as well as extending fluent skills to problem solving. Students who successfully complete Algebra II have a greater likelihood of matriculating and graduating college as well as secure higher starting wages (Adelman, 2006). In response, the Common Core State Standards for Mathematics (2010, CCSS-M) crafted fluency standards originating in kindergarten that extend through middle school (see Table 1).

In order to prepare students for Algebra, NMAP (2008) established three areas of focus termed the Critical Foundations of Algebra: *Fluency with Whole Numbers*, *Fluency with Fractions*, and *Particular Aspects of Geometry and Measurement*. CCCS-M (2010) did not set explicit numerical standards for fluency for simple or complex computation but recommended students focus on developing fluency in writing, interpreting, and translating a variety of linear equations and inequalities, in addition to application in the problem-solving process. The Partnership for Assessment of Readiness for College and Careers Model Content Frameworks for Mathematics (PARCC, 2014) unpacks the

TABLE 1 Core Content State Standards for computational fluency

Gr.	Standard	Description
K	CCSS.MATH.CONTENT.K.OA.A.5	Fluently add and subtract within 5.
1	CCSS.MATH.CONTENT.1.OA.C.6	Add and subtract within 20, demonstrating fluency for addition and subtraction within 10.
	CCSS.MATH.CONTENT.1.NBT.C.5	Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.
2	CCSS.MATH.CONTENT.2.OA.B.2	Fluently add and subtract within 20 using mental strategies. By the end of Grade 2, know from memory all sums of two one-digit numbers.
	CCSS.MATH.CONTENT.2.NBT.B.5	Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
3	CCSS.MATH.CONTENT.3.OA.C.7	Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.
	CCSS.MATH.CONTENT.3.NBT.A.2	Fluently add and subtract within 1,000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
4	CCSS.MATH.CONTENT.4.NBT.B.4	Fluently add and subtract multidigit whole numbers using the standard algorithm.
5	CCSS.MATH.CONTENT.5.NBT.B.5	Fluently multiply multidigit whole numbers using the standard algorithm.
6	CCSS.MATH.CONTENT.6.NS.B.2	Fluently divide multidigit numbers using the standard algorithm.
	CCSS.MATH.CONTENT.6.NS.B.3	Fluently add, subtract, multiply, and divide multidigit decimals using the standard algorithm for each operation.
7	CCSS.MATH.CONTENT.7.EE.B.4.A	Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently.

standards providing additional information and recommendations regarding fluency. To sum up fluency, PARCC (2014) asserts:

> *… (students need to) get past the need to manage computational details so that they can observe structure and patterns in problems. Such fluency can also allow for smooth progress beyond the college and career readiness threshold toward readiness for further study/careers in science, technology engineering, and mathematics (STEM) fields. (pp. 41–42)*

Although initiatives, standards, and prior researchers have established the significance of mathematical fluency, the quality and quantity of practice that occurs in the classroom often fails to promote fluency (NMAP, 2008). In a prior study completed for NMAP (2008), a sample of Algebra teachers (n = 748) conveyed that students need more basic skills preparation in areas such as fractions and decimals, order of operations, and positive and negative integers (Hoffer, Venkataraman, Hedberg, & Shagle, 2007). The teachers also preferred students relying less on calculators. Teachers further suggest that "careful attention to pre-algebra curriculum and instruction in the elementary grades is needed" to remedy skill deficits (p. 35).

Several reasons plausibly explain the gap in research to practice. Ginsburg, Leinwand, Anstrom, and Pollock (2005) suggest that U.S. textbooks present a wide range of topics with less in-depth study versus world leaders in mathematics achievement such as Singapore who spend more time on a smaller number of topics in order to attain mastery. Hence, schools do not produce mathematics proficiency because of the pressure teachers have to move students to new complex, or compound, skills before they can execute the element skills fluently (Binder, 1996; Daly, Martens, Barnett, Witt, & Olson, 2007).

Other researchers stress that textbooks and curricular materials do not appropriately support fluency instruction (Riccomini & Witzel, 2010; Witzel & Riccomini, 2007). Commercial resources frequently contain extraneous materials designed to expand market share. Materials do not contain the appropriate stimuli where students can make effective discriminations; examples may also contain too much information for students to systematically and successfully work through the steps to solve a problem (Daly et al., 2007; Vargas, 1984). As a consequence, students stay in the acquisition phase of learning and either skip to a new topic prematurely or do not invest the appropriate amount of time to engage in systematic practice to reach a level of fluency (Binder, 1996).

1.1 | Fluency defined

Descriptors of fluency include words such as smooth, rhythmic, effortless, fluid, well-practiced, automatic, and masterful (Binder, 1996, Johnson & Layng, 1996; Kubina & Yurich, 2012). Performing with accuracy and speed often serves as the common, plain-language definition for fluency (Binder, 1996). Reaching levels of computational and procedural fluency require systematic practice as does mastering fundamental mathematics vocabulary. Research asserts that fluent execution of mathematical skills reinforces conceptual understanding in order for students to successfully navigate through the mathematics curriculum (NMAP, 2008; PARCC, 2014). Yet some researchers and educators associate practice and fluency with rote memorization, thereby, missing the critical link between conceptual understanding and arithmetical dexterity (Baroody, 2006; Biancarosa & Shanley, 2016; Clarke, Nelson, & Shanley, 2016; NCTM, 2014).

1.2 | Precision teaching and behavioral fluency

Precision Teaching (PT) functions as a measurement system designed to assess behavior and facilitate decision making (Kubina & Yurich, 2012). PT includes pinpointing target behavior, assigning performance criteria, assessing the performance with dimensional units (e.g., frequency, duration, latency), displaying data on a Standard Celeration Chart, and evaluating and making decisions (Lindsley, 1995, 1998; White, 2005). Within the context of applying PT to measure the performance of thousands of students, observations of students performing at high levels of fluency has led to the construct of *behavioral fluency* (Binder, 1996; Kubina & Morrison, 2000).

Behavioral fluency differs from nontechnical uses of "fluency" as described previously. In the seminal article, *Behavioral Fluency: Evolution of a New Paradigm*, Binder (1996) expounded on the concept of *behavioral fluency* and outlined three critical learning outcomes (retention, endurance, and application) that occur after a student reaches a performance standard or criteria (see Table 2). The PT literature suggests students who attain behavioral fluency through precise and well-defined numerical criteria on element skills, experienced smoother transitions to compound skills, and functioned more efficiently and effectively in their natural environments (Binder, 1996; Johnson & Layng, 1996). At the time, the bulk of evidence for behavioral fluency propagated in the PT field was generally communicated in schools, clinical settings, "chart-shares," and in the *Journal of Precision Teaching* (Binder, 1996; Lindsley, 1990; Ramey et al., 2016). Retention, endurance, and application served as evolving terminologies and a call for furthering behavioral fluency research (Binder, 1996).

Frequency building is the most pervasive practice used in PT to build fluency or reach behavioral fluency. Unlike fluency building interventions designed to include only one timed rehearsal with or without feedback, *frequency building* consists of consecutive timed trials with feedback provided after each instance of practice. Students may take an additional timed assessment without feedback to track progress (Kubina & Yurich, 2012). The metric used to calculate frequency of a timed performance involves count over the recorded time of an observed behavior (Johnston & Pennypacker, 2009). Student performance in mathematics typically appear as digits correct per minute (DCPM) or correct problems per minute. All performances examined as frequency provide a continuum of human behavior evaluated on a scale from nonfluent to fluent (Lindsley, 1998).

Subsequent reviews of behavioral fluency offered mixed findings. One review suggested that little experimental evidence exists to substantiate "rate-building procedures" as the causative agent in promoting newer terminologies within the construct of behavioral fluency and field of experimental analysis of behavior (Doughty, Chase, & O'Shields, 2004). Although the Doughty et al.'s (2004) review received criticism for improperly defining behavioral fluency terms, misclassifying studies, and including articles outside the scope of PT (Binder, 2004; Kubina, 2005), the review does focus on the need for methodological rigor for behavioral fluency research. A second review (Ramey et al., 2016) found research on behavioral fluency as an "emerging treatment" for students with developmental disabilities in accordance with the National Autism Center's National Standards Report guidelines. Like the Doughty et al.'s (2004) review, Ramey and colleagues suggest future research in behavioral fluency employ more rigorous research designs to better isolate the effects of critical learning outcomes and numerical criteria that represent behavioral fluency.

TABLE 2 Behavioral fluency: Critical learning outcomes

Critical outcome	Definition	Source of definition
Retention	The relationship between behavior frequencies measured at two points in when a learner has not had the opportunity to perform the behavior	Fabrizio and Moors (2003) Kubina, Amato, Schwilk, and Therrien (2008)
Endurance	The capacity for behavior to occur at a certain speed and accuracy level over extended periods of time; also increases resistance to distraction	Binder, Haughton, and Van Eyk (1990) Kubina and Yurich (2012) Fabrizio and Moors (2003)
Application	The capacity to easily apply the skill as a prerequisite or component of a more complex performance	Johnson and Layng (1996) Kubina and Morrison (2000) Kubina and Yurich (2012)
Maintenance	The relationship between a behavior's frequencies maintained across multiple points in time	Binder (1996) Johnson and Street (2013).
Stability	The predictability of performance over multiple points in time; maintenance of performance in the presence of distractors	Binder (1996) Johnson and Layng (1996) Lindsley (1995)

1.3 | Fluency criteria research

Researchers have noted a paucity of empirical research exists for establishing numerical criteria and measures for fluency in mathematics (Bramlett, Cates, Savina, & Lauinger, 2010; Burns, VanDerHeyden, & Jiban, 2006; Skinner, 2010; VanDerHeyden & Burns, 2008). Published recommendations first generated in the PT literature from the 1970s have limited empirical evidence that rely on small sample sizes (i.e., Deno & Mirkin, 1977; Haughton, 1972; Wood, Burke, Kunzelmann, & Koenig, 1978). Currently, most curriculum-based measures refer to fluency criteria that originated from Deno and Mirkin (i.e., 40 DCPM), which falls significantly below the frequencies of criteria typically recommended (e.g., 80–100 DCPM; 100–110 DCPM) in the PT literature that best lead to behavioral fluency (Haughton, 1972; Johnson & Layng, 1992; Kubina & Yurich, 2012). Burns, VanDerHeyden, and Jiban (2006); VanDerHeyden and Burns, (2005); VanDerHeyden and Burns (2008, 2009) conducted empirical research to establish instructional levels and functional performance criteria for computations. Their findings support 60 DCPM as the criteria for successfully engaging the mathematics curriculum.

With the critical need for mathematics fluency, and recent research showing behavioral fluency outcomes (e.g., McTiernan, Holloway, Healy, & Hogan, 2016; Strømgren, Berg-Mortensen, & Tangen, 2014), we believe that a focused review can contribute to the critical appraisals of the evidence.

1.4 | The present review

Since the special issue in 1996, a database of mathematics intervention studies has emerged detailing critical learning outcomes—retention, endurance, application, maintenance, and stability (see terms and descriptions in Table 2) that occur after students reach a "performance standard" through frequency building. The aim of the present review is to provide an update in behavioral fluency research in mathematics by locating empirical studies that show evidence of critical learning outcomes that occur after a student has reached a fluency performance standard in mathematics. In order to conduct the review, the researchers posed the following questions:

1. What critical learning outcomes occur from students after attaining a performance standard in mathematics?
2. Does the attainment of a performance standard or fluency criteria have associated critical learning outcome(s) resulting from the effects of mathematics interventions?
3. What evidence of fluency criteria best supports behavioral fluency and mathematics achievement?

2 | METHOD

The following criteria led to the inclusion of an article: (a) assessment of a particular skill in mathematics, (b) an identified performance standard, (c) examination of a critical learning outcome of fluency (i.e., retention, endurance, application, maintenance, and stability), (d) reported as an empirical investigation, and (e) appeared in a peer-refereed journal. Search procedures included a computerized search of PsycINFO, ERIC, and ProQuest databases as well as Google Scholar from 1996 to 2018. Descriptors for the studies included the following word combinations: *fluency* and *mathematics*, *fluency* and *computation*, *mathematics fluency* and *retention*, *mathematics fluency* and *endurance*, *mathematics fluency* and *application*, *mathematics fluency* and *maintenance*, and *mathematics fluency* and *stability*.

A hand search of articles contained in the *Journal of Precision Teaching* was conducted. The *Journal of Precision Teaching* has historically functioned as a primary source for disseminating behavioral fluency research. Additionally, an ancestral search of reference lists from the articles determined by the above processes and related review articles (Codding, Burns, & Lukito, 2011; Joseph et al., 2012) identified additional studies.

The initial search of online databases yielded six articles that met the inclusion criteria. An ancestral search of the six articles and two pertinent literature reviews generated three additional articles and an additional five articles resulted from a hand search of the *Journal of Precision Teaching*. The qualifying 14 articles are noted with an asterisk in the reference section.

3 | RESULTS

The results are organized into four sections—retention, application, maintenance, and multiple critical outcomes. The multiple critical outcomes section consists of studies that investigated more than one critical learning outcome. Table 3 provides a summary of the studies conducted including critical learning outcome(s), and referenced performance standards for further review.

3.1 | Retention

VanDerHeyden and Burns (2009) applied a year-long intervention designed to increase fluency with simple computation for students in grades two through five to establish grade-level fluency criteria leading to long-term retention. Students received the practice intervention via class wide peer tutoring 4 days per week with a changing criterion to gradually reach pre-established computational skills goals suggested by Deno and Mirkin (1977).

Each week, the students completed two weekly probes: one assessing the present simple computation skill and the other a mixed skill probe assessing retention of prior weekly skills. Students also took a monthly progress mixed-skill probe for computation to monitor for year-end goals. Results from the investigation showed that students who exhibited retention required significantly higher fluency frequencies than recommended by Deno and Mirkin (1977). VanDerHeyden and Burns (2009) report 60 DCPM for fourth- and fifth-grade students may better support retention and mathematics achievement.

TABLE 3 Critical learning outcomes and performance indicators

Reference	R	E	A	M	S	Performance indicator
Berens, Boyce, Berens, Doney, & Kenzer (2003)	x	x	x			R, M—65 CPPM; E—65 DCPM; A—90 CRPM
Beverley et al. (2009)	x		x			40–60 CPPM with up to 2 errors
Bullara et al. (1993)	x					70–90 DCPM
Chiesa, M., & Robertson, A. (2000)			x			40–50 correct responses per minute
Codding, Eckert, Fanning, Shiyko, & Solomon (2006)			x	x		40 DCPM (Deno & Mirkin, 1977)
Fitzgerald and Garcia (2006)			x			80–100 Correct Responses per minute
Lin and Kubina (2005)			x			80–120 DCPM element; 40–60 DCPM compound
McDowell and Keenan (2002)			x			70–90 DCPM; 80–100 dots per minute 60–80 dots per minute
McTiernan et al. (2016)		x	x		x	50–70 CPPM
MacDonald et al. (2006)				x		60–90 CPPM
Mong and Mong (2010)			x	x		32 DCPM (Deno & Mirkin, 1977)
Singer-Dudek and Greer (2005)				x		100 DCPM
Strømgren et al. (2014)		x			x	70 CPPM
VanDerHeyden and Burns (2009)	x					Grades 2 and 3: 20 DCPM Grades 4 and 5: 40 DCPM

Note. CRPM: correct responses per minute; DCPM: digits correct per minute; R: retention; E: endurance; A: application; M: maintenance; S: stability; CPPM: correct problems per minute.

In an effort to increase addition fact fluency performance of a student receiving special education services at a tutoring clinic, Bullara, Kimball, and Cooper (1993) applied frequency building with flash cards and practice sheets. Intervention components included assessment for accuracy, review of math rules, sprints (15, 30, or 45 s), and at least two 30-s timings for the dependent variable. The participant took a pretest performing at 20 digits correctly in 30 s but could not maintain that pace for a full minute. After attaining the performance standard of 70–90 (74) correct written responses in 1 min, the school recessed for summer break. Three months later in the fall, the researchers conducted a retention check. The participant exhibited a significant degree of retention by writing 56 correct digits per minute.

The practice intervention *Cover, Copy, and Compare* (CCC) has shown to increase accuracy and fluency in multiple subjects such as reading, spelling, geography, and mathematics. Skinner, Bamberg, Smith, and Powell (1993) conducted a study that modified CCC by withdrawing the written component and substituting it with a subvocal component. The participants had to reach a performance standard of 40 DCPM with no errors on three separate sets containing 12 division math facts in each. Participants had to meet the goal on the first set before moving onto the second set. All three participants met the performance standard during the intervention phase as well as maintained the standard 8 months later on retention assessments.

3.2 | Application

Six studies focused solely on application of element skill(s) to a compound skill. Beverley, Hughes, and Hastings (2009) conducted a study of 55 psychology undergraduates exhibiting difficulty with statistics coursework. Drawing on the results from a pretest, 24 students participated a PT flashcards intervention (i.e., *SAFMEDS*, *Say All Fast Minute Everyday Shuffled*) condition and 31 students received treatment as usual in the control condition. The SAFMEDS cards consisted of key statistical terms from the textbook such as bimodal (as related to distribution) and square root (as a related to standard deviation and variance). Fluency criteria was 40–60 correct responses per minute with an allowance of two errors. Participants attended biweekly meetings to review charted progress and receive instructional guidance. Results indicated that students in the experimental condition scored higher score than the control condition on all of the statistics tests throughout the semester. The students in the experimental condition also achieved a statistically significant gain in posttest performance when compared with the control group, $F(1, 53) = 5.23$, $p = 0.026$, $d = 0.62$.

Chiesa and Robertson (2000) conducted a study in a fifth-grade classroom with five learning support students. The balance of the class (20 students) served as the control. Participants practiced component skills (1–5 times tables) including (a) multiplication sheets emphasizing a particular multiplier, (b) finding the missing factor, (c) reciting facts with a peer, and (d) number writing. Students completed a 1-min assessment for each component skill. The researchers set the performance standard of 40–50 correct responses (which included one and two-digit responses) per minute. Participants took pretests and posttests of simple (2 digit by 1 digit without remainders) and more complex (2 digit by 1 digit with remainders) division problems with divisors ranging from ÷ 1 to ÷ 5. After 12 weeks, the results showed that the fluency training group outperformed the control group with the exception of one student. Only 50% of the control group exhibited any gain in performance with 35% scoring lower on the final assessment.

Successful entrance and completion of postsecondary education often depends on prospective undergraduates having a firm grasp of algebra skills (Adelman, 2006). Fitzgerald and Garcia (2006) led a study to improve the performance of five undergraduate students in a remedial algebra course that functioned as a prerequisite for college level algebra coursework. Seven undergraduates in the same course functioned as the control. Presented in 12 sessions over a time period of 6 to 9 weeks, the interventions consisted two to eight 1-min timings designed to increase math fact fluency in all four operations. Although only one participant met the performance criteria of 80–100 correct responses in one operation and the group only averaging an increase of 6.1 DCPM, the implementation of a fluency building intervention proved successful as the experimental group outperformed the matched controls by a mean of 15% in quizzes, exams, and course grades.

In an investigation to examine the relationship between fluency in component to composite skills with simple to complex computation, Lin and Kubina (2005) screened 157 fifth-grade students using explicit timing in a (3) 1-min assessment sequence. The researchers measured (a) writing speed, (b) accuracy and fluency with multiplication facts, and (c) accuracy and fluency with multidigit multiplication problems. Performance standards based on the PT literature included 80–120 DCPM performance standard for multiplication facts (element skill) and 40–60 DCPM for multidigit multiplication (compound skill). Only 14% met the standard for simple computation and 3.2% met the standard for complex computation. The study reported a mean accuracy for component and composite skills at 98.1% and 88.40%, respectively. From the results of the investigation that included a high correlation between element and compound skill fluency (r = 0.745, p = <0.01), Lin and Kubina (2005) determined a majority of the students who did not reach the fluency criteria on simple and complex computation will likely experience difficulties transitioning to new and more complex calculation skills in later coursework.

The PT literature indicates that students who have reached a fluent level of performance on element skills make easier transitions when learning compound skills (Binder, 1996; Johnson & Layng, 1996; Kubina & Yurich, 2012). McDowell and Keenan (2002) designed a study to evaluate the outcome of frequency building with element skills prior to the compound skill versus frequency building with the compound skill prior to the first element then the second element skills. Researchers assigned participants to either +2, ×2, and ×4 written or pointing to keywords as compound skills. Depending on the participant, element skills included (a) written answers to +1, +2, and + 4 problems (criteria of 70–90 DCPM); (b) written dots on a number line to skip count by +2 (criteria of 80–100 dots per minute) or +4 (criteria of 60–80 dots per minute); and/or (c) reciting numbers (60–80 numbers per minute), letter sounds (60–80 sounds per minute). The study reported the seven participants made the most gains across all skills from fluency training starting with element skills versus fluency training starting with compound skills.

3.3 | Maintenance

In order to increase multiplication facts fluency in adults with schizophrenia, MacDonald, Wilder, and Binder (2006) incorporated four phases within the study—baseline, accuracy, fluency, and follow-up (maintenance). During baseline phase, the researchers assigned participants one of 10 worksheets that had 80 problems. After exhibiting stability, the accuracy phase began where participants had to reach 100% accuracy (successfully responding within 5 s of introducing the fact) on 10 sets of facts. The participants then started the fluency practice phase. Practice is composed of goal setting, prompting, verbal feedback, and 1-min timings for two larger combined sets of facts. For the dependent variable, each participant completed a 15-s assessment at the end of each session. To attain a level of fluency, the participants had to answer correctly 60–90 written problems transformed from the 15-s timing. Both reached the fluency criteria. The follow-up phase tested for maintenance occurring once a week for 4 weeks. Results showed that both participants maintained the fluency criteria for the duration of the study.

Mong and Mong (2010) compared the performance of two practice interventions: Cover, Copy, and Compare (a self-managed intervention) and Math to Mastery (teacher mediated) to evaluate which intervention best increased the fluent performance of three second graders with addition and subtraction math facts. Both interventions included the following components—modeling, practice, immediate feedback, and reinforcement. The researchers used an alternating treatments design counterbalancing the interventions on a daily basis. A fluency criteria of 32 DCPM indicated mastery for second-grade students (adapted from Deno & Mirkin, 1977).

After practicing with Cover, Copy, and Compare, the participants completed a 2-min CBM probe. After practicing with Math to Mastery, the participants completed 1-min timings until they reached the fluency criteria and then completed a 2-min CBM probe. Results indicate that Math to Mastery increased student performance more rapidly than Cover, Copy, and Compare. Follow-up data to test for maintenance occurred 6 days and 18 days after the last day of the intervention. Results for maintenance showed that the participants maintained the fluency criterion.

In two separate experiments, Singer-Dudek and Greer (2005) focused on the maintenance of compound skills. The researchers assigned participants to either a mastery learning condition where participants practiced to 100% accuracy or a fluency instruction condition where the participants practiced to a fluency aim of 100 DCPM on math facts while learning complex computation. In the first experiment, the participants practiced simple computation and the multiplication of two-digit by two-digit numbers. In the second experiment, the participants practiced simple computation to the division of three-digit numbers evenly by either the numbers two or three. Results from both experiments demonstrated that participants learned the compound skills in a similar measure of time; however, students who practiced to fluency maintained the compound skill after a 2-month period.

3.4 | Multiple critical outcomes

3.4.1 | Endurance, stability, and application

McTiernan et al. (2016) conducted a random control experiment to evaluate the effects of fluency building on five critical learning outcomes. Prior to intervention both groups of students completed the WIAT-II (Weschler, 2005), a norm-referenced standardized test of mathematical achievement. Both groups also participated in pretest and posttest measures designed to evaluate fluency, endurance, stability, and application. During intervention, students had to orally recite individual sets of fact families within 4 to 6 s each to meet the fluency building criteria. On separate written assessments, fluency aims ranged from 50 to 70 correct written responses on 26 target tasks.

All 26 fluency aims were reached with 71% of participants. The balance of the participants met between 13 and 24 fluency aims. A statistically significant difference occurred between the two groups on the WIAT-II (Weschler, 2005) mathematical reasoning subtest with the experimental group scoring higher, $t(13) = 2.52$, $p = 0.025$, $n_p^2 = 0.333$, versus the control group exhibiting no significant difference, $t(13) = 3.43$, $p = 0.211$, from pretest to post-test. Students in the experimental group also exhibited significant improvement in stability and endurance measures versus the control group. For stability, participants solved paper and pencil practice sheets for 3 min next to students completing problems aloud.

No statistically significant difference occurred between the experimental and control groups in relation to application although the mean score of students in the experimental group yielded 13.8 correct responses on complex computation problems versus 11.1 correct responses for the students in the control group on a 2-min assessment.

3.4.2 | Retention, endurance, and application

Berens, Boyce, Berens, Doney, and Kenzer (2003) conducted three separate experiments that concentrated on retention, endurance, and application. The first study focused on retention from practice with simple computation using flash cards. Results suggest that the participants who emitted vocal responses closer to the fluency aim of 65 correct responses performed better on the retention probe assigned 1 month later. For the study of endurance, the participants vocally identified Arabic numbers in a specific place value. As with retention, the closer the participant performed to the fluency aim (65 correct responses per minute), the better the participant performed on the critical outcome—a 5-min test of endurance.

In the third study that tested for application, the participants vocally identified place values starting at a lower level of difficulty (e.g., ones to tens) and then applied the skills to a higher level of difficulty (e.g., ones up to millions). During training of the target skill, the students also completed application probes at the next level of performance. The results showed that as the participants came closer to the fluency aim of 90 correct responses per minute for the target skill, the participants likewise showed improved performance on higher level application probes that they had yet to receive instruction.

3.4.3 | Application and maintenance

Codding, Eckert, Fanning, Shiyko, and Solomon (2007) evaluated the effects of applying simple computation to complex computation with multiplication and the maintenance of the assigned math facts using three different iterations of the cover, copy, and compare practice intervention: (a) cover, copy, and compare; (b) cover, copy, and compare plus performance feedback using DCPM; and (c) cover, copy, and compare plus feedback using digits incorrect per minute. Assessments for simple computation did not necessarily match the same problems practiced during the intervention.

Participants all met the noted performance standard of 40 DCPM (Deno & Mirkin, 1977) with participants reported reaching as high as 70 DCPM, 57 DCPM, and 43 DCPM. Two participants applied multiplication math facts to three-digit by one-digit with regrouping multiplication problems, and one participant applied addition facts to three-digit by two-digit with regrouping addition problems. Codding et al. (2007) assessed application through a pretest and posttest of complex computation. Two participants completed maintenance checks 4 days and 12 days after the intervention whereas one participant completed one maintenance check 4 days after the intervention. Results showed that all three participants maintained basic computation skills and had small increases in complex computation fluency.

3.4.4 | Endurance and stability

In an 8-week study of fifth-, sixth-, and seventh-grade students, Strømgren et al. (2014) tested for endurance and stability with math facts using fluency criteria of 70 correct responses per minute. Participants in the experimental group received a PT frequency building intervention. Practice included 30-s timings with feedback for a maximum of 10 timings per day. Endurance tests lasted 90 s. Stability tests lasted 30 s while students listened to music in headphones as a distractor. Although most students only reached between 55 and 65 in the experimental condition, Strømgren et al. (2014) reported that participants who scored above 50 correct responses per minute in the experimental condition maintained scores more reliably on the stability and endurance assessments.

4 | DISCUSSION

The present review examined frequency building and practice interventions used to increase behavioral fluency in mathematics. Behavioral fluency denotes a student's ability to (a) *retain* information for later retrieval, (b) *endure* and complete tasks, (c) *apply* fluent skills to new learning scenarios, (d) *maintain* a level of fluent performance after achieving a standard or benchmark, and (e) exhibit a *stable* or predictable performance in the face of distraction—all critical learning outcomes that can support successful participation in the mathematics curriculum. Studies that qualified for the review had to include a predetermined fluency criterion and a measurement of one or more of the associated critical learning outcomes. The studies that met the inclusion criteria applied a wide range of fluency criteria starting at 20–40 to upwards of 60–100 correct responses or DCPM.

By establishing fluency criteria and intermediate goal setting, teachers can effectively prepare materials that match a student's instructional level and make informed decisions regarding student progress (Binder, 1996; Daly et al., 2007; Haring & Eaton, 1978; Kubina & Yurich, 2012). Engaging in activities designed to increase fluency helps students to automatically recall number combinations while executing standard algorithms versus using inferior strategies such as finger-counting, tally marks (Carr, Taasoobshirazi, Stroud, & Royer, 2011). Students who are fluent adapt more readily to new and more complex material introduced by the teacher and have less trouble keeping up with pace of instruction (Clarke et al., 2016; NMAP, 2008).

The PT literature makes a distinction between retention and maintenance. Retention measures performance after an extended period-of-time without intervention (e.g., 1 month or more) whereas maintenance entails measuring student performance on a set number of occasions (e.g., daily, weekly, and biweekly) within closer proximity to the last day of intervention (Kubina & Yurich, 2012). Students who reach higher frequencies in performance increase the probability of long-term retention (Binder, 1996; Fabrizio & Moors, 2003). In the eight studies that measured maintenance or retention, participants sustained gains in performance regardless of fluency criteria. Although the studies report 60-plus digits or responses correct per minute best supported retention and maintenance, the studies did not investigate the criteria that delineates the numerical threshold that separates both terms.

As students increase in frequency of responding, they typically build endurance and stability—the capacity to stay on task for an extended period-of-time and not engage in competing stimuli that previously delivered higher levels of reinforcement (Kubina & Yurich, 2012). Theorists have attributed the increase in frequency or the repetitive, continual movement and associated reinforcement analogous to building momentum necessary to complete the practice sequence in the face of competing stimuli or "distraction" (Banda, Matuszny, & Therrien, 2009; Lee, 2006; Nevin, Mandell, & Atak, 1983). When frequency of accurate responses increases, so does rate of reinforcement providing added benefits such as assignment completion and enhanced academic performance (Banda & Kubina, 2010; Lee, 2006). The process of building endurance and stability can plausibly alleviate fatigue and propel students to "not give up" or quit on element and compound skills as well as show more interest in the activity (Binder et al., 1990; Brady & Kubina, 2010).

Criteria and assessments varied between the three studies that measured endurance and/or stability. Strømgren et al. (2014) indicated that students perform better when emitting more than 50 correct responses per minute on 90-s written endurance and stability assessments for simple computation. Berens et al. (2003) indicated 100 oral responses per minute on 5-min endurance assessments for identifying numbers (Berens et al., 2003). McTiernan et al. (2016) did not provide a specific numerical recommendation for endurance and stability. Still, the mean response of the treatment group that included 29% of students who did not reach full criteria averaged 40 DCPM over the 5-min endurance measure and 45.5 DCPM on the 3-min stability measure.

Research provides evidence on the deleterious effects of nonfluent performance in mathematics, starting as early as at the preschool and primary school levels and continuing throughout school careers (e.g., Duncan et al., 2007; Geary, 2013; Geary, Hoard, Nugent, & Bailey, 2013). To maintain proficient performance in mathematics, a student must successfully engage early in the curriculum where element or easier skills precede compound, or more complex skills. Students who have difficulty solving elements (e.g., computing decimals and fractions) often continue to struggle when practicing new compounds (e.g., order of operations; Binder, 1996; Johnson & Layng, 1992; Kubina & Yurich, 2012). Findings in eight out of the nine studies suggest that increasing the frequency of correct responding led to better performance outcomes on application assessments; however, it is important to note that five out of the nine studies controlled for element skill correspondence in next-level compound skills whereas four studies tested for overall performance in curriculum or coursework.

4.1 | Recommendations for future research

Behavioral fluency occurs when a student reaches a performance standard or fluency criteria. Yet inconsistencies still exist in the literature that link numerical fluency indicators to grade-level performance in mathematics. It is likely that national initiatives such as NMAP (2008), NCTM (2000), and CCCSS (2010) have not recommended numerical standards due to a lack of empirical evidence. The PT literature recommends a range between 80 DCPM and 110 DCPM for simple computation (Haughton, 1972; Johnson & Layng, 1996; Kubina & Yurich, 2012; Wood et al., 1978). Deno and Mirkin (1977) recommended 40 DCPM originating from a PT program in a Minnesota school that has persisted in subsequent work in curriculum-based measurement (Shapiro, 2004; Shinn, 1989). Limited data exist to support either reference point as a valid, empirically derived measure (Burns et al., 2006).

Burns, VanDerHeyden, and colleagues (2006, 2008, 2009) conducted initial research to validate fluency criteria recommended by Deno and Mirkin (1977) and the PT literature. Findings suggest that useful retention will likely occur closer to 60 DCPM. Future research validating grade-level or developmentally appropriate fluency criteria using standardized measures can provide a firm reference point for investigating the subsequent, intricate measures associated with critical learning outcomes. Without establishing empirically validated fluency criteria of minimum performance for element and compound skills, practitioners may continue to have limited knowledge and confidence in implementing the instructional methods designed to meet fluency standards. Hence, moving the field of behavioral fluency research forward requires systematic replication and extended investigation into establishing functional numerical criteria.

Maintenance or response maintenance is the accepted term for the extent in which the student continues to perform a target behavior after the intervention has been terminated (Cooper, Heron, & Heward, 2007). The PT literature suggests that students who emit higher frequencies of a behavior are more likely to demonstrate retention (long term) versus maintenance (short term; Kubina & Yurich, 2012). Despite the increase in retention and maintenance, a paucity of research exists to define and then systematically isolate and operationalize the minimum criteria that separate the terms. Establishing numerical criteria to differentiate between the two terms can provide an important contribution to behavioral research such as long-term and short-term memory have in cognitive and educational psychology (Cowan, 2008). For teachers, matching the appropriate criteria to reach levels of maintenance and retention can play an important role in formative and summative assessment. Teachers can prioritize skill sets, intensify practice to meet goals, and modify instructional materials to optimize student progress (Barnett, Daly, Jones, & Lentz, 2004; Daly et al., 2007; Haring & Eaton, 1978).

The PT literature also differentiates between application and generalization. Precision teachers use the term application to measure whether students have directly combined fluent element skills into a compound skill. Generalization can occur if students reach fluency with element skills but can also occur in novel instances and environments (Binder, 2003). As noted, six of the nine studies investigated application with elements and compounds whereas three studies used application out of the behavioral fluency context to assess the generalizable effects of frequency building on mixed-skill assessments and standardized tests. Accurate, systematic, and operationalized use of terms related to principles of behavior (e.g., Baer, Wolf, & Risley, 1968) can provide more validity to behavioral fluency research.

When investigating application, the effects researchers seek may not occur immediately when students first apply the fluent element skills to the compound skill. A compound skill requires systematic practice to reach procedural fluency. For instance, a student who fluently executes simple computation often has the advantage of building speed and accuracy when engaging the long division algorithm. A student who is nonfluent with simple computation has to allocate cognitive resources towards inferior strategies (e.g., finger counting) when engaging the new procedure (Binder, 1996; Carr et al., 2011; Kubina & Yurich, 2012). The small effect McTiernan et al. (2016) experienced may have occurred from conducting only one assessment testing application of simple to complex computation. Future application studies could benefit from extending days of systematic practice with the compound skill after the experimental group reaches criteria with element skills to determine the effects of fluent and nonfluent element skill performance has on compound skill performance.

5 | CONCLUSION

The studies included in the present review provides insight into interventions that lead to behavioral fluency in mathematics. The review also highlights the discrepancy in criteria within the research literature of what constitutes a fluent performance. As suggested, further research to determine grade-level criteria and the most effective interventions that can help students expeditiously increase behavioral fluency that yield critical learning outcomes. Furthermore, by systematically defining terms, evaluating the components, and isolating their effects through carefully engineered studies that clearly illustrate that experimental control has been achieved, behavioral fluency can further contribute to mathematics education and applied behavior analysis.

CONFLICTS OF INTEREST

The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements) or nonfinancial interest (such as personal or professional relationships, affiliations, knowledge, or beliefs) in the subject matter or materials discussed in this manuscript.

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